Parameters and Order Identification of the Fundamental Linear Fractional Systems of Commensurate Order

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***Abstract*—The identification of fractional order systems is a more difficult problem than the integer one because it requires not only the estimation of the model coefficients but also the determination of the fractional orders with the tedious numerical calculation of the fractional order derivatives. This paper addresses the time domain identification of the dynamical fundamental linear fractional system of commensurate order described by linear fractional order differential equation. The proposed identification technique is based on the recursive least squares algorithm applied to a linear regression equation using adjustable fractional order differentiator to estimate the model's parameters and the commensurate order at the same time. Illustrative examples are also presented to validate the usefulness of the proposed identification approach.**

Keywords-Adjustable fractional order differentiator; Least squares method; Linear fractional differential equation; Recursive identification.

I. INTRODUCTION

Recently, fractional order derivatives and integrals have been used in different domains of science and engineering [1-5]. Hence, the fractional order systems have become an increasingly popular tool for modeling the behavior of many physical systems where the derived models are generally well described by fractional order differential equations rather than classical ones [6-10]. So, their identification is gaining more and more interests from the scientific community in spite of its difficult task because it requires not only the estimation of the model parameters but also the determination of the fractional orders with the tedious numerical calculation of the fractional order derivatives. Some fractional order systems identification techniques have been proposed in the literature where most of them are extensions of identification methods for the regular integer order systems and require a priori knowledge of the fractional differentiation orders and estimate only the model's parameters [11-15].

In [15], the proposed identification technique uses the recursive least squares algorithm to estimate the parameters, with a priori knowledge of the fractional differentiation orders, through a regression equation derived by the discretization of the fractional order linear differential equation using a digital adjustable fractional order differentiator. This paper addresses the time domain identification of the fundamental linear fractional system of commensurate order. The proposed identification method is an extension of the work of [15] to include the estimation of the commensurate order besides of the estimation of the model parameters. The basic ideas and the derived formulations of the identification scheme are presented. Illustrative examples are also presented to validate the proposed identification approach of the fundamental linear fractional system of commensurate order.

II. PRELIMINARIES

A. *Fundamental linear fractional system of commensurate order*

The fundamental linear single input single output (SISO) fractional system of commensurate order is described by the following linear fractional order differential equation [17]:

$$y(t) + aD^{m} y(t) = be(t) \tag{1}$$

where e(t) is the input, y(t) is the output, m is the commensurate order ($0 < m < 1$), the model parameters a and b are constant real numbers. With zero initial conditions, the fundamental fractional system of (1) transfer function is :

$$G(s) = \frac{Y(s)}{E(s)} = \frac{b}{1 + as^{m}} \tag{2}$$

B. *Digital FIR Adjustable Fractional Order Differentiator*

The analog fractional order differentiator can be approximated, in a given frequency band of interest [ω_L, ω_H], by a rational transfer function as follows [17]:

$$G_D(s) = s^{m} \cong K_D + \sum_{i=0}^{N_D} \frac{k_{Di} s}{\left(1 + s/p_{Di}\right)}, \quad 0 < m < 1 \tag{3}$$

The poles p_{Di} ($0 \le i \le N_D$) are calculated only once for m = 0.5. The coefficient K_D and the residue k_{Di} ($0 \le i \le N_D$) can be interpolated as $K_D(m)=\sum_{n=0}^{M_D} d_n m^n$ $\quad k_{Di}(m)=\sum_{n=0}^{M_D} d_{in} m^n$; where d_{in} ($0 \le i \le N_D$ and $0 \le n \le M_D$) and d_n ($0 \le n \le M_D$) are independent of the fractional order m [17]. Hence, the analog adjustable fractional order differentiator transfer function is given by:

$$G_D(s)=s^m \cong \sum_{n=0}^{M_D} m^n \left[d_n + \sum_{i=0}^{N_D} \frac{d_{in} s}{\left(1+s/p_{Di}\right)} \right] = \sum_{n=0}^{M_D} m^n G_{Dn}(s) \quad (4)$$

Using the Euler generating function $s=\frac{1-z^{-1}}{T}$ (T is the sampling period), the digital FIR adjustable fractional order differentiator transfer function $G_D(z)$ is then derived from (4) as follows [15]:

$$s^m \cong G_D(z)=\sum_{n=0}^{M_D} m^n \left\{ \sum_{k=0}^{L-1} g_{Dn}(k)\, z^{-k} \right\} \quad (5)$$

where,

$$g_{Dn}(k)=d_n\delta(k)+\sum_{i=0}^{N_D}\left(\frac{d_{in}p_{Di}}{Tp_{Di}+1}\right)\left\{\left[\frac{1}{Tp_{Di}+1}\right]^{k} u(k)-\left[\frac{1}{Tp_{Di}+1}\right]^{k-1} u(k-1)\right\}.$$

So, the proposed digital definition of the fractional differentiator of order 0 < m < 1 is given as:

$$D^m y(t=kT) \cong \sum_{n=0}^{M_D} m^n \left[\sum_{q=0}^{L-1} g_{Dn}(q)\, y(k-q) \right] \quad (6)$$

In this case, we note that the term $\left[\sum_{q=0}^{L-1} g_{Dn}(q)\, y(k-q) \right]$ of (6) is completely in dependant of the fractional differentiator order m. Hence, it is calculated only one time; besides, the fractional differentiation $D^m y(t=kT)$ can be easily and quickly obtained for different values of m without repeating the tedious calculations of the fractional differentiations for each value of m.

III. MODEL IDENTIFICATION METHOD

A. Problem formulation

In this context, we will consider the development of an identification technique for the class of dynamic linear SISO fundamental fractional systems of commensurate order represented by the fractional differential equation of (1); where the commensurate order m, the parameters a and b are unknown; then they have to be subject to parametric estimation. Using the digital the fractional order derivative of (6), the fractional differential equation of (1), considered to be initially at rest, is converted to a discrete-time form as:

$$y(k)+a\sum_{n=0}^{M_D} m^n \left[\sum_{q=0}^{L-1} g_{Dn}(q)\, y(k-q) \right] = be(k) \quad (7)$$

Rearranging (7) for q = 0 and $1 \le q \le (L-1)$, we will get:

$$y(k)\left[1+\left\{a\sum_{n=0}^{M_D} m^n g_{Dn}(0)\right\}\right]+a\sum_{n=0}^{M_D} m^n\left[\sum_{q=1}^{L-1} g_{Dn}(q)\,y(k-q)\right]=be(k) \quad (8)$$

Hence, the solution y(k) of the fractional differential equation of (1) is given as:

$$y(k)=-\frac{a\sum_{n=0}^{M_D} m^n\left[\sum_{q=1}^{L-1} g_{Dn}(q)\,y(k-q)\right]}{\left[1+\left\{a\sum_{n=0}^{M_D} m^n g_{Dn}(0)\right\}\right]}+\frac{be(k)}{\left[1+\left\{a\sum_{n=0}^{M_D} m^n g_{Dn}(0)\right\}\right]} \quad (9)$$

A simplified form of (9) is given by the following expression:

$$y(k)=-a'Y(k)+b'U(k) \quad (10)$$

where a', $Y(k)$, b' and $U(k)$ are:

$$a'=\frac{a}{\left[1+\left\{a\sum_{n=0}^{M_D} m^n g_{Dn}(0)\right\}\right]},$$
$$b'=\frac{b}{\left[1+\left\{a\sum_{n=0}^{M_D} m^n g_{Dn}(0)\right\}\right]} \quad (11)$$
$$Y(k)=\sum_{n=0}^{M_D} m^n\left[\sum_{q=1}^{L-1} g_{Dn}(q)\,y(k-q)\right], \qquad U(k)=e(k)$$

Then, the system output y(k) can be expressed as the following regression form:

$$y(k)=\phi(k)\theta=\left[Y(k) \quad U(k)\right]\begin{bmatrix} a' \\ b' \end{bmatrix} \quad (12)$$

The regression equation of (12) is a structure which is linear in the unknown parameter vector $\theta=\left[a' \quad b'\right]^T$; hence it can be solved using the classical least squares approach.

B. Parameters estimation step

From (12), the estimated solution $\hat{y}(k)$ is given by the structure model:

$$\hat{y}(k)=\phi(k)\hat{\theta}=\left[Y(k) \quad U(k)\right]\begin{bmatrix} \hat{a}' \\ \hat{b}' \end{bmatrix} \quad (13)$$

where the parameter vector $\theta=\left[\hat{a}' \quad \hat{b}'\right]^T$ is unknown and the numerical values of the function vector $\phi(k)=\left[Y(k) \quad U(k)\right]$ are calculated from the observations y(k) and u(k) using the last two expressions of (11). Yet, the function vector $\phi(k)$ is known only if the estimated commensurate order $\hat{m}$ is given. In this proposed approach the initial estimated commensurate order $\hat{m}$ is given. In the classical least squares approach, the vector $\hat{\theta}$ is selected in order to minimize the quadratic least squares criterion [18-19]:

$$J(\theta)=\frac{1}{2}\left[\sum_{k=1}^{Km}\left[y(k)-\hat{y}(k,\theta)\right]^2\right] \quad (14)$$

The recursive estimated parameter vector $\hat{\theta}$ using the above quadratic least squares criterion is given by [18-19]:

$$\begin{cases} \hat{\theta}(k+1)=\hat{\theta}(k)+\left\{F(k)\,\phi(k+1)\,\mathrm{E}r(k+1)\right\} \\ F(k+1)=\frac{1}{\lambda}\left[F(k)-\left\{\frac{F(k)\phi(k+1)\phi^T(k+1)F(k)}{1+\phi^T(k+1)F(k)\phi(k+1)}\right\}\right] \\ \mathrm{E}r(k+1)=\frac{y(k+1)-\left\{\hat{\theta}^T(k)\phi(k+1)\right\}}{\lambda+\phi^T(k+1)F(k)\phi(k+1)} \end{cases} \quad (15)$$

where $0<\lambda<1$ and the initial value F(0) of the adaptation gain matrix F(k) is $F(0)=\frac{1}{\delta}I$ with $(0<\delta<<1)$.

C. Commensurate order estimation step

From (10), the estimated solution $\hat{y}(k)$ is given by:

$$\hat{y}(k)=-\hat{a}'Y(k)+\hat{b}'U(k) \quad (16)$$

The estimated parameter vector $\hat{\theta}=\left[\hat{a}' \quad \hat{b}'\right]^T$ is known from the current iteration of the parameters estimation step, and from (11) the expression $Y(k)$ is given as follows:

$$Y(k)=\sum_{n=0}^{M_D}\hat{m}^n\left[\sum_{q=1}^{L-1}g_{Dn}(q)y(k-q)\right] \quad (17)$$

where $\hat{m}$, the estimated commensurate order used in the current iteration of the parameters estimation step, is obtained from the previous iteration of the commensurate order estimation step. So, the estimation error Er(k) of the current iteration of the parameters estimation step is:

$$\mathrm{E}r(k)=y(k)-\hat{y}(k)=y(k)-\left[-\hat{a}'Y(k)+\hat{b}'U(k)\right] \quad (18)$$

In this context, the estimated commensurate order $\hat{\mathrm{m}}$ is adjusted by the steepest descent algorithm so that the square error $\left[\mathrm{Er(k)}\right]^2$ is minimized. Then, the updating equation of $\hat{\mathrm{m}}$ is given by:

$$\hat{m}(k+1)=\hat{m}(k)-\mu\frac{d\left[\mathrm{E}r(k)\right]^2}{d\hat{m}}=\hat{m}(k)-2\mu Er(k)\,g(k) \quad (19)$$

where μ is the step size and $\mathrm{g(k)}=\frac{\mathrm{d}\left[\mathrm{Er(k)}\right]}{\mathrm{d}\hat{\mathrm{m}}}$ is the gradient of Er(k). Because y(k) does not depend on the commensurate order $\hat{\mathrm{m}}$, the function g(k) is given as:

$$g(k)=\frac{d\left[\mathrm{E}r(k)\right]}{d\hat{m}}=\frac{d\left[y(k)-\hat{y}(k)\right]}{d\hat{m}}=-\frac{d\left[\hat{y}(k)\right]}{d\hat{m}} \quad (20)$$

$$g(k)=-\frac{d}{d\hat{m}}\left\{-\hat{a}'Y(k)+\hat{b}'U(k)\right\}=\left\{\hat{a}'\frac{dY(k)}{d\hat{m}}-\hat{b}'\frac{dU(k)}{d\hat{m}}\right\} \quad (21)$$

Because y(k), e(k) and g_{Dn}(k) do not depend on the commensurate order $\hat{m}$, the expressions $\frac{dY(k)}{d\hat{m}}$ and $\frac{dU(k)}{d\hat{m}}$ can be easily calculated as:

$$\frac{dY(k)}{d\hat{m}}=\frac{d}{d\hat{m}}\left\{\sum_{n=0}^{M_D}\hat{m}^n\left[\sum_{q=1}^{L-1}g_{Dn}(q)y(k-q)\right]\right\} \; and \; \frac{dU(k)}{d\hat{m}}=\frac{de(k)}{d\hat{m}}$$

$$\frac{dY(k)}{d\hat{m}}=\frac{d}{d\hat{m}}\left\{\sum_{n=0}^{M_D}n(\hat{m}^{(n-1)})\left[\sum_{q=1}^{L-1}g_{Dn}(q)y(k-q)\right]\right\} \; and \; \frac{dU(k)}{d\hat{m}}=0$$

Once the estimated parameters vector $\hat{\theta}=\left[\hat{a}' \quad \hat{b}'\right]^T$ and the estimated commensurate order $\hat{m}$ are calculated, the estimated parameters $\hat{a}$ and $\hat{b}$ of the coefficients a and b of the fractional differential equation of (1) are then derived from the first two expressions of (11) as follows:

$$\hat{a}=\frac{\hat{a}'}{\left[1-\left\{\hat{a}'\sum_{n=0}^{M_D}\hat{m}^n g_{Dn}(0)\right\}\right]} \text{ and } \hat{b}=\frac{\hat{b}'}{\left[1-\left\{\hat{a}'\sum_{n=0}^{M_D}\hat{m}^n g_{Dn}(0)\right\}\right]} \quad (23)$$

D. Algorithm

1) Step 1

- The word "data" is plural, not singular. Set the parameter δ , λ , μ and ε such that $0<\delta<<1$, $0<\lambda<1$, $0<\mu<1$ and $0<\varepsilon<<1$.
- Set the initial value $F(0)=\frac{1}{\delta}I$ of the adaptation gain matrix F(k).
- Set the initial value of the estimated parameter vector $\hat{\theta}(0)=\left[\hat{a}'=0 \quad \hat{b}'=0\right]^T$.
- Set the initial value of the estimated commensurate order $\hat{m}(0)=m_0\neq 0$.
- Given the input e(k) and its corresponding observation y(k) samples, compute the term $\left[\sum_{q=1}^{L-1}g_{Dn}(q)y(k-q)\right]$ of Y(k) in (11) (this term is computed only one time).

2) *Step 2*

- Use $\hat{m}(k)$ to compute $Y(k)$ of (11).
- Use (15) to compute $\hat{\theta}(k+1)$.

3) *Step 3*

- Use (18) to compute Er(k).
- If the mean square error $\frac{\sum [\mathrm{Er(k)}]^2}{K} < \varepsilon$; stop the iterations (K is the number of samples).
- Otherwise, use (22) to compute the gradient g(k) of (21).
- Use (19) to compute $\hat{m}(k+1)$.
- Set k = k+1 and got step 2.

IV. Illustrative Examples

To validate the effectiveness of the proposed identification approach of linear fractional system of commensurate order an illustrative example is presented. The considered example is a stable fundamental fractional system of commensurate order represented by the following linear fractional order differential equation:

$$y(t) + 0.1\frac{d^{0.57}y(t)}{dt^{0.57}} = 2.0e(t) \qquad (24)$$

The observations data y(t) which will be used in the identification algorithm are calculated using the Grunwald-Letnikov fractional differentiation definition from the linear fractional order differential equation of (24) as follows:

$$y(t=kT) = \frac{-\left\{\frac{0.1}{T^{0.57}}\sum_{q=1}^{K} w_q^{(0.57)} y(kT-qT)\right\}}{\left(1+\frac{0.1}{T^{0.57}}\right)} + \frac{\{2.0e(kT)\}}{\left(1+\frac{0.1}{T^{0.57}}\right)} \qquad (25)$$

where $w_q^{0.57} = (-1)^q \dfrac{\Gamma(0.57+1)}{\Gamma(q+1)\Gamma(0.57-q+1)}$.

The parameters used for the numerical calculation in this example are:

- Sampling period T = 0.1s
- Length of the Grunwald-Letnikov fractional differentiation K = 100
- Number of iterations of the observation horizon N_H = 200
- Digital FIR adjustable fractional order differentiator parameters
  - ✓ N_D = 35
  - ✓ M_D = 10
  - ✓ Length of the impulse response L = 150

A. *First case: Simulation with a unit step input:*

The proposed identification fractional model is represented by the following linear fractional order differential equation:

$$y(t) + a\frac{d^m y(t)}{dt^m} = be(t) \qquad (26)$$

In this case, the input e(k) is a unit step and the observation samples y(k) are obtained from (25).

The recursive least squares algorithm of (15) for the parameter estimation is initialized as follows:

$\delta = 10^{-4}$, $F(0) = \frac{1}{\delta}I$, $\lambda = 0.98$ and $\hat{\theta}(0) = [a'(0)=0, b'(0)=0]$

and the recursive least squares algorithm of (19) for the commensurate order estimation is initialized as follows:

$\mu = 10^{-1}$ and $\hat{m}(0) = 0.98 > 0.57$

Figures (1) and (2) show, respectively, the evolution of the estimated parameters and the commensurate order of the fractional model of (26).

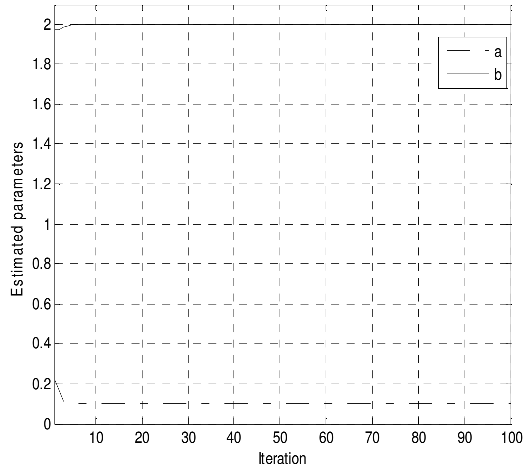

Figure 1. Estimated parameters of the fractional model

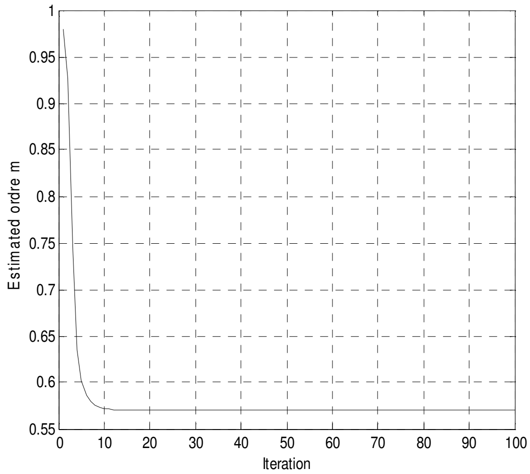

Figure 2. Estimated commensurate order $\hat{m}$ of the fractional model

Table (1) shows the estimated values of the parameters and the commensurate order of the fractional model of (26).

TABLE 1. ESTIMATED VALUES OF THE PARAMETERS AND THE COMMENSURATE ORDER

Parameter	m	a	b	Estimation error
Exact value	0.57	0.1	2.0	
Estimated value	0,5703	0,1000	2,0000	2,1270e-09

Now the recursive least squares algorithm of (19) for the commensurate order estimation is initialized as follows:

$$\mu = 10^{-1} \text{ and } \hat{m}(0) = 0.15 < 0.57$$

Figures (3) and (4) show, respectively, the evolution of the estimated parameters and the commensurate order of the fractional model of (26).

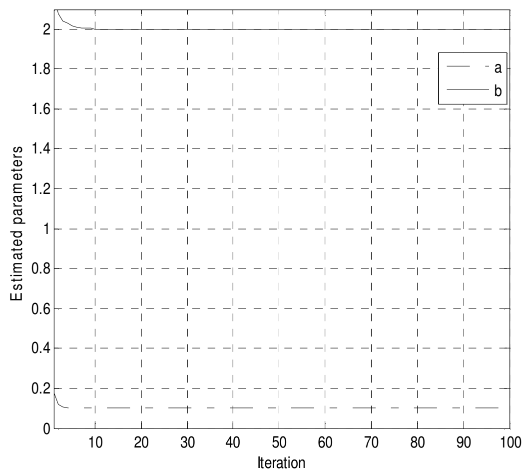

Figure3. Estimated parameters of the fractional model

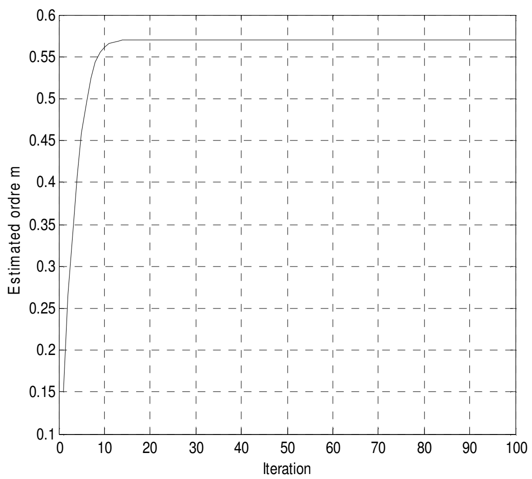

Figure 4. Estimated commensurate order $\hat{m}$ of the fractional model

Table (2) shows the estimated values of the parameters and the commensurate order of the fractional model of (26).

TABLE 2. ESTIMATED VALUES OF THE PARAMETERS AND THE COMMENSURATE ORDER

Parameter	m	a	b	Estimation error
Exact value	0.57	0.1	2.0	
Estimated value	0,5703	0,1000	2,0000	2,1270e-09

We remark that for different initializations ($\hat{m}(0) = 0.98$ and $\hat{m}(0) = 0.15$) of the recursive least squares algorithm of (19) for the commensurate order estimation, the estimated parameters and the fractional order of the fractional model of (26) are all the same.

B. Second case: Simulation with a pseudo random binary sequence (PRBS) input

For this case, the observation samples y(k) are also obtained from (25) for the input e(k) a pseudo random binary sequence. The recursive least squares algorithm of (15) for the parameter estimation is initialized as follows:

$$\delta = 10^{-4}, F(0) = \frac{1}{\delta} I, \lambda = 0.98 \text{ and } \hat{\theta}(0) = [\hat{a}(0) = 0, \hat{b}(0) = 0]$$

and the recursive least squares algorithm of (19) for the commensurate order estimation is initialized as follows:

$$\mu = 10^{-1} \text{ and } \hat{m}(0) = 0.98 > 0.57$$

The evolution of the estimated parameters and the commensurate order of the fractional model of (26) are shown, respectively, in figures (5) and (6).

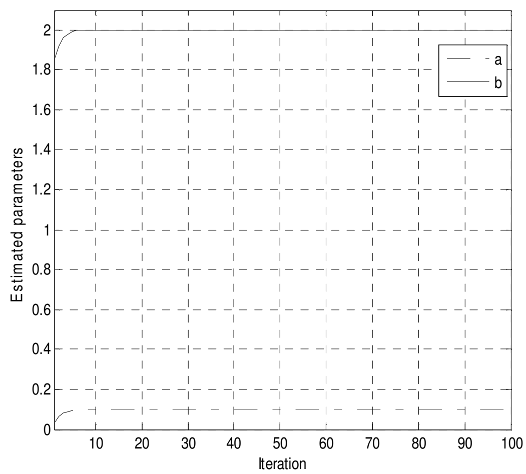

Figure 5. Estimated parameters of the fractional model

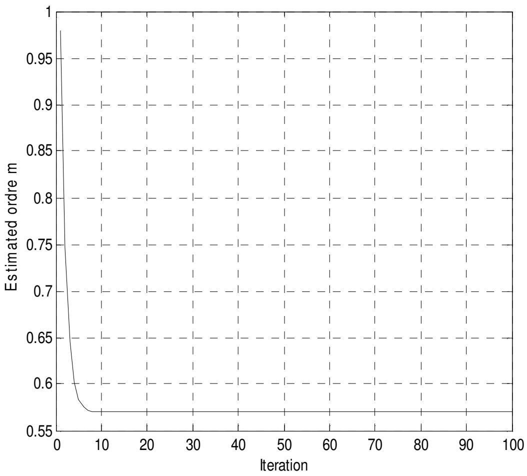

Figure 6. Estimated commensurate order $\hat{m}$ of the fractional model

The exact and the estimated values of the parameters and the commensurate order of the fractional model of (26) are given in Table (3).

TABLE **3**. ESTIMATED VALUES OF THE PARAMETERS AND THE COMMENSURATE ORDER

Parameter	**m**	**a**	**b**	**Estimation error**
Exact value	0.57	0.1	2.0	
Estimated value	0,5699	0,1000	2,0000	4,1943e-11

Now let us initialize the recursive least squares algorithm of (19) for the commensurate order estimation as follows:

$$\mu = 10^{-1} \text{ and } \hat{m}(0) = 0.15 < 0.57$$

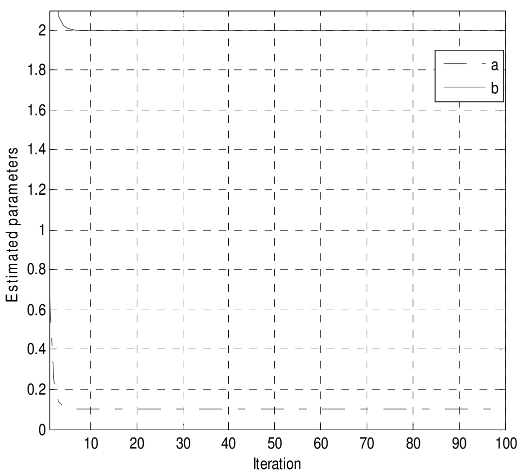

Figure 7. Estimated parameters of the fractional model

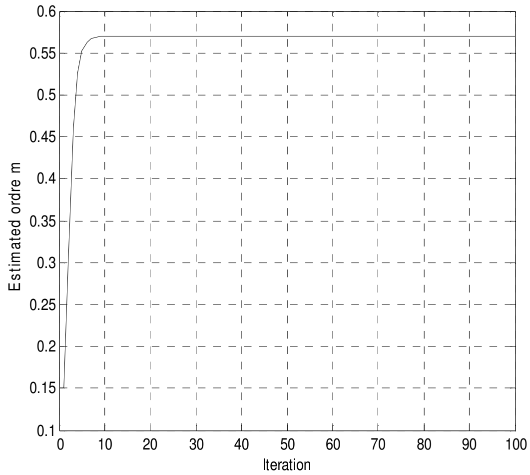

Figure 8. Estimated commensurate order $\hat{m}$ of the fractional model

The exact and the estimated values of the parameters and the commensurate order of the fractional model of (26) are given in Table (4).

TABLE **4**. ESTIMATED VALUES OF THE PARAMETERS AND THE COMMENSURATE ORDER

Parameter	**m**	**a**	**b**	**Estimation error**
Exact value	0.57	0.1	2.0	
Estimated value	0,5699	0,1000	2,0000	4,1943e-11

For this type of input, we also remark that for different initializations ($\hat{m}(0) = 0.98$ and $\hat{m}(0) = 0.15$) of the recursive least squares algorithm of (19) for the commensurate order estimation, the estimated parameters and the fractional order of the fractional model of (26) are all the same.

V. CONCLUSION

In this paper, a time domain identification method of the fundamental linear fractional system of commensurate order has been proposed. This identification scheme is based on the recursive least squares algorithm applied to a linear regression equation obtained by using an adjustable fractional order differentiator to estimate the model's parameters and the commensurate order at the same time. Numerical simulations for different inputs and different initializations of the commensurate order have been presented to validate the effectiveness of the proposed identification method. The obtained results were very satisfactory. The basic idea of the proposed identification scheme may be extended to higher fractional system of commensurate order.

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