A simple Isoparametric Finite Element Based on the Reddy's Theory for the Laminated Plates Bending Analysis

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***Abstract*— The aim of this work is to develop a quadrilateral finite element based on Reddy's third order shear deformation theory for the bending behavior analysis of composite laminated plates. The element is a C0 four-nodded isoparametric with seven degrees of freedom at each node, three translation components, two rotations and two higher order rotational degrees. In particular, selective numerical integration is introduced in order to improve the results and to alleviate the locking phenomenon. The performance and reliability of the proposed formulation are demonstrated by comparing the author's results with those obtained using the three-dimensional elasticity theory, analytical solutions and other advanced finite element models.**

Keywords*— *Third Order Shear Deformation Theory; Laminated Composite Plates; Finite Element; Bending Behavior;

I. Introduction

Nowadays, the multilayer composite materials have found increasingly wide applications essentially in all industrial sectors, such as aerospace, automotive, civil engineering and shipbuilding. This considerable use is probably due to the remarkable benefits of this type of materials namely; an excellent rigidity weight, good corrosion resistance, fatigue resistance and many more advantages particularly since their properties are adjustable for different situations. A review was published by Mouritz et al. [1] for the recent applications of composite structures and their developments on the ships and submarines. On the other hand, the analysis of multilayer composite structures is still questionable and soliciting accurate theories about complicated to describe their behavior and different mechanical phenomena. For example, modeling thick multilayer structures requires refined theories taking into account good expression of the effect due to the transverse shear deformation through their thicknesses and in particular the interlaminar. Therefore, several theories take into account the transverse shear effect on the analysis of multilayer composite structures that are the extension of the equivalent single layer approach [2]. The classical laminated plate theory (CLPT) [3, 4] is one of the oldest and simplest theories in describing the deformation of the plates. However, it is adequate for the analysis thin structures because of the negligence of the effect due to the transverse shear. While, the simplest theory which takes into account the deformation of transverse shear is the first order shear deformation theory [FSDT] [5, 6]. However, these theories require to correction factors [6-9]. Consequently, theories of higher order shear deformation (HSDT) have been proposed in the literature [10, 11], in accurately assessing the deformation and stress of transverse shear multilayer plates without to need factors correction. Moreover, the theory of Reddy (TSDT) is the higher order theory most frequent for multilayer plates analysis when it is able to assess the stresses and transverse shear deformations with a small number unknown and do not depend on the layer number [12, 13]. However, the third order theory of Reddy [TSDT] encounters a problem when finites elements are applied by requiring the second derivative C_1 [14], it is identical in the finite elements development of thin plate based on the classical theory [15]. Therefore, many of finite element models (2D) based on Reddy's third order theory were proposed in the literature and in various geometries and nodes, and thus different number of degrees of freedom [16] for the analysis the static behavior of laminated composite plates. A review has published by Zhang and Yang (2009) [16] contains a recent development finite element for the analysis of laminated composite plates.

The objective of this paper is the development a new simple finite element less expensive in terms of accuracy and stability on the basis of Reddy's third order theory (TSDT) by adopting the approach of equivalent single layer, and able to analyze the bending behavior of isotropic and laminated composite plates by ensuring the right compromise between the cost and precision.

II. Kinemtic

The displacement field according the Reddy's third order shear deformation theory (TSDT) [12] can be expressed as follows:

$$
\begin{aligned}
u_1 &= u + z\psi_x - \frac{4z^3}{3h}(\psi_x + \frac{\partial w}{\partial x}),\\
u_2 &= v + z\psi_y - \frac{4z^3}{3h}(\psi_y + \frac{\partial w}{\partial y}),\\
u_3 &= w
\end{aligned}
\qquad (1)
$$

Where u, v, w are the displacements to the median plane of the plate and ψ_x, ψ_y are rotations about the axes y and x respectively, and h is the thickness of the plate. The deformations associated with displacement field (1), given as follows:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_3}{\partial x}\right)^2\right] = \varepsilon_1^0 + z\left(\kappa_1^0 + z^2\kappa_1^2\right);$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u_3}{\partial y}\right)^2\right] = \varepsilon_2^0 + z\left(\kappa_2^0 + z^2\kappa_2^2\right);$$

$$\varepsilon_{23} = \left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} + \frac{\partial u_3}{\partial y}\frac{\partial u_3}{\partial z}\right) = \varepsilon_4^0 + z^2\kappa_4^2;$$

$$\varepsilon_{13} = \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial z}\right) = \varepsilon_5^0 + z^2\kappa_5^2;$$

$$\varepsilon_{12} = \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y}\right) = \varepsilon_6^0 + z\left(\kappa_6^0 + z^2\kappa_6^2\right); \qquad (2)$$

III. THE CONSTITUTIVE EQUATIONS

The stress-strain relationships laminated to the k-ième layer after the global coordinate x, y, z transformation [17] and according the transformation matrix and the stress-strain relationships, it is given by:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}_k = \begin{bmatrix}\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16}\\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26}\\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66}\end{bmatrix}_k \begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} \qquad (3)$$

$$\begin{Bmatrix}\sigma_{yz}\\ \sigma_{xz}\end{Bmatrix}_k = \begin{bmatrix}\bar{C}_{44} & \bar{C}_{45}\\ \bar{C}_{45} & \bar{C}_{55}\end{bmatrix}_k \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}$$

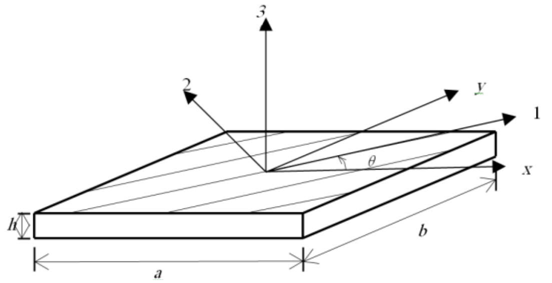

Fig. 1 Geometry and coordinate system of the laminated composite plate

IV. VIRTUAL WORK PRINCIPLE

The static equations of the theory can be derived from the virtual work principle [12] by expressing the strain variation energy as follows:

$$\int_{-h/2}^{h/2}\int_A \left(\sigma_1\delta\varepsilon_1 + \sigma_2\delta\varepsilon_2 + \sigma_6\delta\varepsilon_6 + \sigma_5\delta\varepsilon_5 + \sigma_4\delta\varepsilon_4\right) dA dz + \int_A q\delta W dA = 0 \qquad (4)$$

According to the substitution of equations (2) in the static equation (4) we obtain:

$$\int_A [N_1\frac{\partial\delta u}{\partial x} + M_1\frac{\partial\delta\psi_x}{\partial x} + P_1(-\frac{4}{3h^2}(\frac{\partial\delta\psi_x}{\partial x} + \frac{\partial^2\delta w}{\partial x^2})) + N_2\frac{\partial\delta v}{\partial y} + M_2\frac{\partial\delta\psi_y}{\partial y} +$$
$$P_2(-\frac{4}{3h^2}(\frac{\partial\delta\psi_y}{\partial y} + \frac{\partial^2\delta w}{\partial y^2})) + N_6(\frac{\partial\delta u}{\partial y} + \frac{\partial\delta v}{\partial x}) + M_6(\frac{\partial\delta\psi_x}{\partial y} + \frac{\partial\delta\psi_y}{\partial x}) + \qquad (5)$$
$$P_6(-\frac{4}{3h^2}(\frac{\partial\delta\psi_x}{\partial y} + \frac{\partial\delta\psi_y}{\partial x} + 2\frac{\partial^2\delta w}{\partial x\partial y})) + Q_2(\delta\psi_y + \frac{\partial\delta w}{\partial y}) + R_2(-\frac{4}{h^2}(\delta\psi_y + \frac{\partial\delta w}{\partial y}))$$
$$+Q_1(\delta\psi_x + \frac{\partial\delta w}{\partial x}) + R_2(-\frac{4}{h^2}(\delta\psi_x + \frac{\partial\delta w}{\partial x})) + q\delta w]dA = 0$$

Where N_i, M_i, P_i, Q_1, Q_2 and R_1, R_2 are the resulting forces to define:

$$(N_i, M_i, P_i) = \sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\sigma_i\left(1, z, z^3\right)dz; \quad (i = 1, 2, 6);$$

$$(Q_2, R_2) = \sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\sigma_4\left(1, z^2\right)dz; (Q_1, R_1) = \sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\sigma_5\left(1, z^2\right)dz;$$

Therefore, the generalized relations resulting forces can be given as follows [12]:

$$\begin{Bmatrix}\{N_i\}\\ \{M_i\}\\ \{P_i\}\end{Bmatrix} = \begin{bmatrix}[A_{ij}] & [B_{ij}] & [E_{ij}]\\ & [D_{ij}] & [F_{ij}]\\ sym & & [H_{ij}]\end{bmatrix}\begin{Bmatrix}\varepsilon^0\\ \kappa^0\\ \kappa^2\end{Bmatrix}$$

$$\begin{Bmatrix}\begin{Bmatrix}Q_2\\ Q_1\end{Bmatrix}\\ \begin{Bmatrix}R_2\\ R_1\end{Bmatrix}\end{Bmatrix} = \begin{bmatrix}A_{44} & A_{45} & D_{44} & D_{45}\\ & A_{55} & D_{45} & D_{55}\\ & & F_{44} & F_{45}\\ sym & & & F_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon^0{}_4\\ \varepsilon^0{}_5\\ \kappa^2{}_4\\ \kappa^2{}_5\end{Bmatrix} \qquad (6)$$

Where: $\left[\underbrace{\varepsilon^0{}_1 \;\; \varepsilon^0{}_2 \;\; \varepsilon^0{}_6}\;\; \underbrace{\kappa^0{}_1 \;\; \kappa^0{}_2 \;\; \kappa^0{}_6}\;\; \underbrace{\kappa^2{}_1 \;\; \kappa^2{}_2 \;\; \kappa^2{}_6}\right]^T = \left[\varepsilon^0 \;\; \kappa^0 \;\; \kappa^2\right]^T;$

$$\left[\underbrace{\varepsilon^0{}_4 \;\; \varepsilon^0{}_5}\;\; \underbrace{\kappa^2{}_4 \;\; \kappa^2{}_5}\right]^T = \left[\gamma^s \;\; \kappa^s\right]^T$$

Where:

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\bar{C}_{ij}(1, z, z^2, z^3, z^4, z^6)dz, \quad (i, j = 1, 2, 6)$$

$$(A_{ij}, D_{ij}, F_{ij}) = \sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\bar{C}_{ij}(1, z^2, z^4)dz, \quad (i, j = 4, 5)$$

Substituting the resulting forces defined in equation (5), we obtain:

$$\int_A(\delta\varepsilon^{0T}[A]\varepsilon^0 + \delta\varepsilon^{0T}[B]\kappa^0 + \delta\varepsilon^{0T}[E]\kappa^2 + \delta\kappa^{0T}[B]\varepsilon^0 + \delta\kappa^{0T}[D]\kappa^0$$
$$+\delta\kappa^{0T}[F]\kappa^2 + \delta\kappa^{2T}[E]\varepsilon^0 + \delta\kappa^{2T}[F]\kappa^0 + \delta\kappa^{2T}[H]\kappa^2 + \delta\gamma^{sT}[A^s]\gamma^s$$
$$+\delta\gamma^{sT}[D^s]\kappa^s + \delta\kappa^{sT}[D^s]\gamma^s + \delta\kappa^{sT}[F^s]\kappa^s + \varepsilon^{\circ T}{}_{NL}\{N\} + q\delta w)dA = 0 \qquad (7)$$

V. FINITE ELEMENT FORMULATION

The present finite element is a C_0 four nodes isoparametric having seven DOF for each node (Fig. 2). Three displacements *u, v, w*, two rotations ψ_x, ψ_y and two higher order rotations θ_x, θ_y where $\theta_x = (\psi_x + \partial w/\partial x)$, $\theta_y = (\psi_y + \partial w/\partial y)$. The analytical integration can be converted to Gauss's numerical integration [18]. For more model performance in terms of accuracy and stability and to avoid shear locking problem, the technical selective numerical integration is used from Gauss points (2×2) for membrane and flexional contribution and (1×1) point for the transverse shear contribution.

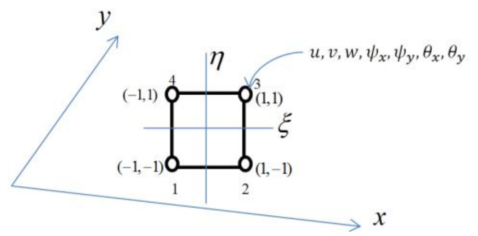

Fig. 2 Description of the normalized isoparametric element

A. *Nodal approximation*

The nodal approximation is expressed from the Lagrange interpolation con-cerning the considered degrees of freedom in each node and the element geometry through all coordinates (ζ, η), the field variables may be expressed as follows:

$$u(\xi,\eta)=\sum_{i=1}^{4}\bar{N}_i(\xi,\eta)u_i;\; v(\xi,\eta)=\sum_{i=1}^{4}\bar{N}_i(\xi,\eta)v_i;\; w(\xi,\eta)=\sum_{i=1}^{4}\bar{N}_i(\xi,\eta)w_i;$$

$$\psi_x(\xi,\eta)=\sum_{i=1}^{4}\bar{N}_i(\xi,\eta)\psi_{xi};\; \psi_y(\xi,\eta)=\sum_{i=1}^{4}\bar{N}_i(\xi,\eta)\psi_{yi};\; \theta_x(\xi,\eta)=\sum_{i=1}^{4}\bar{N}_i(\xi,\eta)\theta_{xi};$$

$$\theta_y(\xi,\eta)=\sum_{i=1}^{4}\bar{N}_i(\xi,\eta)\theta_{yi};\; x=\sum_{i=1}^{4}\bar{N}_i(\xi,\eta)x_i;\; y=\sum_{i=1}^{4}\bar{N}_i(\xi,\eta)y_i,$$

$$\text{Where}: \bar{N}_i(\xi,\eta)=\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$$

Ni(ζ,η) are the bi linear interpolation functions of Lagrange type corresponding to node i=1,2,3,4 [19].

B. *Deformation and nodal displacements relation*

The deformation vectors and the nodal unknowns can be taken as the elementary nodal matrix forms as follows:

$$\varepsilon^0=\left[B_\varepsilon^0\right]^T\{\delta\}=\begin{bmatrix}\frac{\partial\bar{N}_i}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{\partial\bar{N}_i}{\partial y} & 0 & 0 & 0 & 0 & 0\\ \frac{\partial\bar{N}_i}{\partial y} & \frac{\partial\bar{N}_i}{\partial x} & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ w_i\\ \psi_{x\,i}\\ \psi_{y\,i}\\ \theta_{x\,i}\\ \theta_{y\,i}\end{Bmatrix}$$

$$\kappa^0=\left[B_\kappa^0\right]^T\{\delta\}=\begin{bmatrix}0 & 0 & 0 & \frac{\partial\bar{N}_i}{\partial x} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{\partial\bar{N}_i}{\partial y} & 0 & 0\\ 0 & 0 & 0 & \frac{\partial\bar{N}_i}{\partial y} & \frac{\partial\bar{N}_i}{\partial x} & 0 & 0\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ w_i\\ \psi_{x\,i}\\ \psi_{y\,i}\\ \theta_{x\,i}\\ \theta_{y\,i}\end{Bmatrix}$$

$$\kappa^2=\left[B_\kappa^2\right]^T\{\delta\}=c_1\begin{bmatrix}0 & 0 & 0 & 0 & 0 & \frac{\partial\bar{N}_i}{\partial x} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial\bar{N}_i}{\partial y}\\ 0 & 0 & 0 & 0 & \frac{\partial\bar{N}_i}{\partial y} & \frac{\partial\bar{N}_i}{\partial x} & \end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ w_i\\ \psi_{x\,i}\\ \psi_{y\,i}\\ \theta_{x\,i}\\ \theta_{y\,i}\end{Bmatrix}$$

$$\gamma^s=\left[B_\varepsilon^s\right]^T\{\delta\}=\begin{bmatrix}0 & 0 & \frac{\partial\bar{N}_i}{\partial y} & 0 & \bar{N}_i & 0 & 0\\ 0 & 0 & \frac{\partial\bar{N}_i}{\partial x} & \bar{N}_i & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ w_i\\ \psi_{x\,i}\\ \psi_{y\,i}\\ \theta_{x\,i}\\ \theta_{y\,i}\end{Bmatrix}$$

$$\kappa^s=\left[B_\kappa^s\right]^T\{\delta\}=c_2\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & \bar{N}_i\\ 0 & 0 & 0 & 0 & 0 & \bar{N}_i & 0\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ w_i\\ \psi_{x\,i}\\ \psi_{y\,i}\\ \theta_{x\,i}\\ \theta_{y\,i}\end{Bmatrix}$$

With: $c_1=-4/3h^2$, $c_2=-4/h^2$

According to the substitution of deformation and nodal displacements relations in the equilibrium equation (7) and from the static system $[K]_e\{\delta\}=\{F\}$ we can conclude the elementary stiffness matrix $[K]_e$ as follow:

$$[K]_e=\int_{-1}^{1}\int_{-1}^{1}\Big(\left[B^0{}_\varepsilon\right]^T[A]\left[B^0{}_\varepsilon\right]+\left[B^0{}_\varepsilon\right]^T[B]\left[B^0{}_\kappa\right]+\left[B^0{}_\varepsilon\right]^T[E]\left[B^2{}_\kappa\right]+\left[B^0{}_\kappa\right]^T[B]\left[B^0{}_\varepsilon\right]+$$
$$\left[B^0{}_\kappa\right]^T[D]\left[B^0{}_\kappa\right]+\left[B^0{}_\kappa\right]^T[F]\left[B^2{}_\kappa\right]+\left[B^2{}_\kappa\right]^T[E]\left[B^0{}_\varepsilon\right]+\left[B^2{}_\kappa\right]^T[F]\left[B^0{}_\kappa\right]+\left[B^2{}_\kappa\right]^T[H]\left[B^2{}_\kappa\right]+$$
$$\left[B^s{}_\varepsilon\right]^T\left[A^s\right]\left[B^s{}_\varepsilon\right]+\left[B^s{}_\varepsilon\right]^T\left[D^s\right]\left[B^s{}_\kappa\right]+\left[B^s{}_\kappa\right]^T\left[D^s\right]\left[B^s{}_\varepsilon\right]+\left[B^s{}_\kappa\right]^T\left[F^s\right]\left[B^s{}_\kappa\right]\Big)\det[J]\,d\xi d\eta \quad (8)$$

Where : $\{F\},\{\delta\}$ are the elementary nodal vectors of forces and unknowns displacement, respectively.

VI. RESULTS AND DISCUTION

A. *Convergence study*

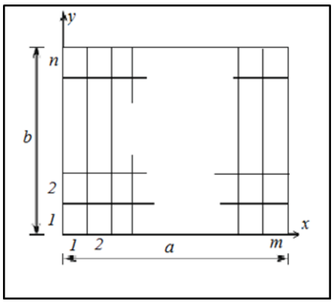

Fig. 3 . ***Regular mesh of rectangular plate m × n***

1) *Exemple 1*

In the first example, the convergence of the developed quadrilateral element is studied for a simply supported isotropic square plate subjected to uniformly distributed load. Two thickness ratios (*a*/*h*=10, 100) are considered for the analysis. The non-dimensional results of transverse displacement, for different mesh sizes are displayed on Table 1.

The non-dimensional transverse displacement value is described from: $\bar{w}=100wD/qa^4$ *where:* $D=Eh^2/12(1-\nu^2)$ with ν=0.3.

The comparison was made with the analytical solutions given by Reddy [14], Kant [20] and Dobyns [21] and the finite element solution given by Sheikh [14], Kant et al.[22] and Goswami and Wilfried [23]. The results of the comparison show the performances and convergence of the present formulation. It can be noticed that the present model is applicable in both thick and thin isotropic plates.

Table. 1 *Convergence test of transverse displacement of an isotropic square plate simply supported under uniform loading.*

TABLE I. CONVERGENCE TEST OF TRANSVERSE DISPLACEMENT OF AN ISOTROPIC SQUARE PLATE SIMPLY SUPPORTED UNDER UNIFORM LOADING.

Reference	Theory	$\bar{w}$	
		a/h=10	*a/h=100*
Present (4×4)	*TSDT*	*0,4245*	*0.4027*
Present (8×8)		*0.4293*	*0.4079*
Present (12×12)		*0.4282*	*0.4071*
Present (16×16)		*0.4278*	*0.4068*
Present (20×20)		*0.4275*	*0.4067*
Reddy [14]	*TSDT*	*0.4273*	*0.4064*
Kant [20]	*HSDT*	*0.4240*	*0.4060*
AL Dobyns[21]	*Analytical Solution*	*0.4237*	*0.4064*
Kant et al.[22]	*HSDT (FE)*	*0.4260*	*0.407*
Sheikh [14]	*HSDT (FE)*	*0.4274*	*0.4066*

Goswami and Wilfried [23]	HSDT (FE)	0.43126	0.40531

2) *Exemple 2*

In this example, we consider a simply supported square three-layer (0/90/0) symmetrical cross-ply laminated plate subjected to a sinusoidal load L1, and properties material $E_1/E_2=25$, $G_{12}=G_{13}=0.5E_2$, $G_{23}=0.2E_2$, $v=0.25$.. The nondimensional central deflexion and maximum stresses of the plate have been determined for different thickness ratio *a/h*. Very satisfactory results have been obtained in Table. 2 for a mesh size of 16×16 comparing to those obtained analytically by the Pagano's 3D Elasticity solution [7, 24], Reddy [12] and Wang and Shi [25] and those obtained by other finite elements models based on different theories as HSDT and the first order FSDT by Sheikh [14] and TSDT layer-wise by Ramesh et al.[26].

TABLE II. *NONDIMENSIONAL DEFLEXION AND STRESSES OF A SQUARE LAMINATED PLATE (0/90/0) UNDER A SINUSOIDAL LOAD*

Reference	*Theory*	$\frac{a}{h}$	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$-\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
Present	*TSQ28*	*4*	*1.936*	*0.7657*	*0.5073*	*0.0495*	*0.2061*	*0.1797*
Pagano [7, 24]	*Elasticity*		*2.0042*	*0.8010*	*0.534*	*0.0505*	*0.256*	*0.2172*
Wang and Shi [25]	*HSDT*		*1.9734*	*0.8439*	*0.5972*	*0.0508*	*0.2409*	*0.2297*
Reddy[12]	*TSDT*		*1.9218*	*0.7345*	-	-	-	*0.1832*
Sheikh [14]	*HSDT(FE)*		*1.923*	*0.7500*	*0.5080*	*0.0499*	*0.2023*	*0.1831*
Sheikh [14]	*FSDT(FE)*		*1.777*	*0.4430*	*0.4843*	*0.0371*	*0.1440*	*0.1569*
Ramesh et al.[26]	*TSDT layer-wise(FE)*		*1.9136*	*0.7672*	*0.5081*	*0.0500*	*0.2809*	*0.2103*
Present	*TSQ28*	*10*	*0.7198*	*0.5833*	*0.2709*	*0.0278*	*0.2610*	*0.1017*
Pagano [7, 24]	*Elasticity*		*0.7405*	*0.5900*	*0.285*	*0.0289*	*0.3570*	*0.1228*
Wang and Shi [25]	*HSDT*		*0.7549*	*0.5946*	*0.2919*	*0.029*	*0.3566*	*0.1239*
Reddy[12]	*TSDT*		*0.7125*	*0.5684*	-	-	-	*0.1033*
Sheikh [14]	*HSDT(FE)*		*0.7140*	*0.5806*	*0.2722*	*0.0279*	*0.2437*	*0.1015*
Sheikh [14]	*FSDT(FE)*		*0.6700*	*0.5219*	*0.2582*	*0.025*	*0.1623*	*0.0918*
Ramesh et al.[26]	*TSDT layer-wise(FE)*		*0.7178*	*0.5850*	*0.2713*	*0.0281*	*0.3671*	*0.1178*
Present	*TSQ28*	*20*	*0.5065*	*0.5492*	*0.2046*	*0.0229*	*0.2745*	*0.0808*
Pagano [7, 24]	*Elasticity*		*0.5142*	*0.552*	*0.210*	*0.0234*	*0.3850*	*0.0938*
Wang and Shi [25]	*HSDT*		*0.5176*	*0.553*	*0.2106*	*0.0234*	*0.3845*	*0.094*
Ramesh et al.[26]	*TSDT layer-wise(FE)*		*0.5060*	*0.5509*	*0.2050*	*0.0231*	*0.3871*	*0.0917*
Present	*TSQ28*	*100*	*0.4344*	*0.5376*	*0.1801*	*0.0211*	*0.2794*	*0.0731*
Pagano [7, 24]	*Elasticity*		*0.4368*	*0.5390*	*0.1810*	*0.0213*	*0.3950*	*0.0828*
Wang and Shi [25]	*HSDT*		*0.4356*	*0.5392*	*0.1808*	*0.0214*	*0.3946*	*0.0829*
Reddy[12]	*TSDT*		*0.4342*	*0.5390*	-	-	-	*0.0750*
Sheikh [14]	*HSDT(FE)*		*0.4350*	*0.5496*	*0.1828*	*0.0215*	*0.2401*	*0.0749*
Sheikh [14]	*FSDT(FE)*		*0.4350*	*0.5490*	*0.1825*	*0.0202*	*0.1568*	*0.0709*
Ramesh et al.[26]	*TSDT layer-wise(FE)*		*0.4345*	*0.5394*	*0.1806*	*0.0214*	*0.3952*	*0.0835*

The nondimensional values are described in Table. 4 as given by:

$$\bar{w} = w(\frac{a}{2},\frac{b}{2},0)(h^3E_2/q_0a^4)10^2, \bar{\sigma}_{xx} = \sigma_{xx}(\frac{a}{2},\frac{b}{2},\frac{h}{2})(h^2/q_0a^2),\ \bar{\sigma}_{yy} = \sigma_{yy}(\frac{a}{2},\frac{b}{2},\frac{h}{6})(h^2/q_0a^2),$$

$$\bar{\sigma}_{xy} = \sigma_{xy}(0,0,\frac{h}{2})(h^2/q_0a^2),\ \bar{\sigma}_{xz} = \sigma_{xz}(0,\frac{b}{2},0)(h/q_0a), \bar{\sigma}_{yz} = \sigma_{yz}(\frac{a}{2},0,0)(h/q_0a)$$

a) *Distrubution of stress thrugh the palte thickness (0/90/0)*

The structure type of symmetric cross-ply plate (0/90/0) is one of the most used in the literature for validation examples. Moreover, this structure has been treated analytical by the Pagano's 3D Elasticity solution [24].

In Figs (5, 6, 7), the curves show good distributions obtained of normal stresses through the thickness of the laminated plate (0/90/0) by comparing to those obtained analytically by Pagano's Elasticity and those obtained by Soldatos theory (HSDPT) [27].

In addition, from Figs (8, 9) the present finite element provides a good parabolic distribution transverse shears stress σ_{yz}, σ_{xz} through the thickness of the plate without to use any correction factors. The results have been compared with those obtained analytically by Pagano's 3D Elasticity solution [24] and Soldatos's theory (HSDPT) [27].

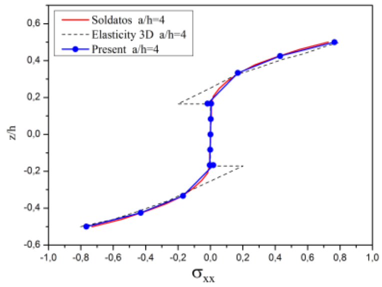

Fig. 4 . *Distribution of the normal stress $\sigma_{x,x}$ through the thickness of a simply supported laminated plate (0/90/0) subjected to a sinusoidal load ratio of a/h = 4*

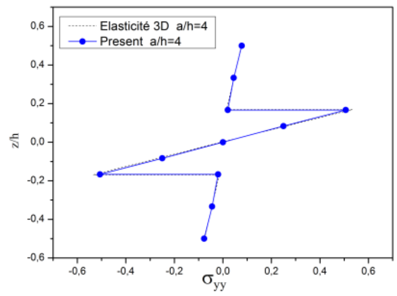

Fig 5. *Distribution of the normal stress σ_{yy} through the thickness of a simply supported laminated plate (0/90/0) subjected to a sinusoidal load ratio of* a/h = 4

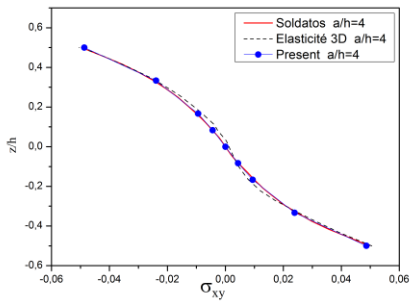

Fig 7. Distribution of the normal stress σ_{xy} through the thickness of a simply supported laminated plate (0/90/0) subjected to a sinusoidal load ratio of **a/h = 4**

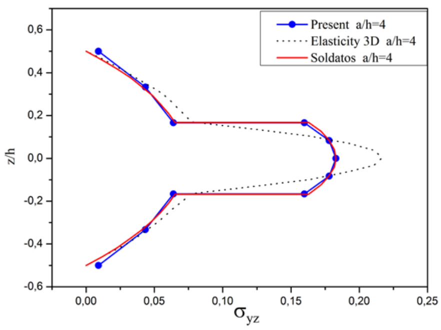

Fig 8. Distribution of the transverse stress σ_{yz} through the thickness of a simply supported laminated plate (0/90/0) subjected to a sinusoidal load ratio of a/h = 4

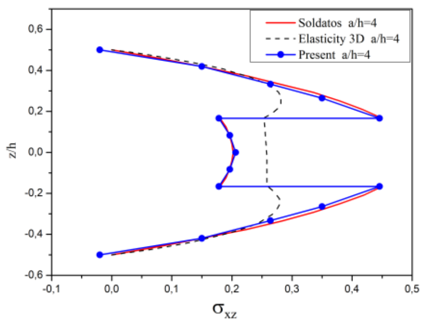

Fig 9. Distribution of the transverse stress σ_{xz} through the thickness of a simply supported laminated plate (0/90/0) subjected to a sinusoidal load ratio of a/h = 4

VII. Conclusion

In this paper, a developed four nodes isoparametric finite element is proposed on the basis on Reddy's third order shear deformation theory by adopting the equivalent single-layer approach. The present element has seven degrees of freedom for each node, three displacements, two rotations and two higher order rotations. The formulation is able to take into account the transverse shear effect in order to analyze the bending behavior of isotropic and laminated composite thin and thick plates without to need correction factors. Furthermore, the selective numerical integration technique is conducted in order to get accurate results without including numerical locking problem.

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